SOLUTION OF THE PLANAR LINEAR FILTRATION PROBLEM IN LAYERED SOILS

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Under natural conditions inhomogeneity in soils is caused by their layered structure [1]. A wide range of literature has considered problems of liquid filtration in layered soils (the interlinking problem) in either exact or approximation formulation (see the overview of [2]). A solution was first obtained in exact formulation for the problem of interlinking of two homogeneous zones separated by a straight line or a circle in [3]. In [4-7] such problems were solved for three or four homogeneous zones separated by circles or parallel lines. The techniques used were the reflection method [4], bipolar coordinates [5], reduction of the problem to a system of integral equations [6], and solution of functional equations [7]. In the present study the Fourier method and conventional functions will be used to obtain a solution for the general planar interlinking problem for an arbitrary number of inhomogeneous zones, separated by parallel lines, where the flow is induced by arbitrary singular points of specified slow growth functions at infinity. In the case of layers with parabolic, in particular, constant, permeability, the potentials in each layer are found in terms of quadratures of harmonic function singularities. The layers may be arbitrarily interleaved with fissures and slightly permeable screens, which are modeled by infinitely thin layers having infinitely high permeability in the case of fissures, or infinitely low for permeability screens.

We will consider linear liquid filtration in a plane x, y, divided into zones $D_i(y_i < y < y_{i-1})$, with permeability functions $K_i(y) \in C^1$, where the flow is induced by singular points of the functions $f_i(x, y)$ in the zones D_i, where $K_i > 0$, $i = 1, \ldots, n + 1$, $y_0 = \infty$, $y_{n+1} = -\infty$, $f_i(x, y_j) = O(|x|^m)$, while the functions $f_i(x, y_j)$ have limits as $|x| \to \infty$. For the potentials ϕ_i having singular points of the functions f_i in the zones D_i we have the interlinking problem:

$$\operatorname{div}(K_i \nabla \varphi_i) = 0, \quad i = 1, \ldots, n+1; \tag{1}$$

$$y = y_i{:}\ \varphi_i = \varphi_{i+1}, \ K_i \partial\varphi_i/\partial y = K_{i+1}\partial\varphi_{i+1}/\partial y, \ i = 1, \ldots, n. \tag{2}$$

In the general case $|f_i| \not\in L(\partial D_i)$, for example, where f_i are the source potentials and therefore the Fourier method is inapplicable.

Considering that differentiation with respect to $x(D_x)$ reduces the order of the poles of the functions $f_i(x, y_j)$ at infinity and that Eqs. (1), (2) are invariant with respect to the operation D_x, we will consider the problem of Eqs. (1), (2) for the functions $u_i = D_x^k\varphi_i$, $k \leqslant m + 2$, which have singular points of the functions $F_i = D_x^k f_i$, while $|F_i| \in L(\partial D_i)$. The operation D_x corresponds to Σ-differentiation of Σ-monogenic functions [8]. Applying the Fourier method, we find the functions u_i in the form

$$u_i = F_i + \frac{1}{\pi}\int_0^\infty \sum_{\nu=1}^{2} \left(a_i^\nu \eta_i + b_i^\nu \xi_i\right)\delta_\nu \, d\lambda, \tag{3}$$

where ν is an index, $a_1^\nu = b_{n+1}^\nu = 0$; $\delta_1 = \cos\lambda x$; $\delta_2 = \sin\lambda x$; $\eta_i(y, \lambda)$, $\xi_i(y, \lambda)$ are solutions, finite in D_i, of the Cauchy problem

$$(K_i\eta_i')' - \lambda^2 K_i\eta_i = 0; \tag{4}$$

$$\begin{aligned} &y = y_1{:}\ \ \xi_1 = 1, \qquad y = y_{i-1}{:}\ \ \xi_i = 0, \ \ \xi_i' = -\lambda, \\ &y = y_n{:}\ \ \eta_{n+1} = 1, \ \ y = y_i{:}\ \ \ \eta_i = 1, \ \ \eta_i' = 0, \end{aligned} \tag{5}$$

$\eta_i' = D_y\eta_i$; the parameters $a_i^\nu(\lambda)$, $b_i^\nu(\lambda)$ satisfy the system of algebraic equations

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$$a_i^\nu + b_i^\nu \xi_{ii} - a_{i+1}^\nu \eta_{i+1i} = r_i^\nu, \tag{6}$$
$$K_{ii} b_i^\nu \xi_{ii}' + K_{i+1i} a_{i+1}^\nu \eta_{i+1i}' - K_{i+1i} b_{i+1}^\nu = R_i^\nu, \quad i = 1, \ldots, n.$$

Here

$$\xi_{ii} = \xi_i > 0, \quad \xi_{ii}' = -\frac{\xi_i'}{\lambda} > 0, \quad \eta_{i+1i} = \eta_i > 0, \quad \eta_{i+1i}' = \frac{\eta_{i+1}'}{\lambda} > 0 \tag{7}$$

for $y = y_i$, $\lambda > 0$; $K_{ij} = K_i(y_j)$; $r_i^\nu = h_{i+1i}^\nu - h_{ii}^\nu$; $R_i^\nu = (K_{ii} H_{ii}^\nu - K_{i+1i} H_{i+1i}^\nu)/\lambda$; h_{ij}^ν and H_{ij}^ν are Fourier coefficients of the functions $F_i(x, y_j)$ and $D_y F_i(x, y_j)$ in $\cos \lambda x$ for $\nu = 1$ and in $\sin \lambda x$ for $\nu = 2$. The inequalities of Eq. (7) are satisfied because of the properties of the solutions of equations of the form of Eq. (4): if $\eta_i \eta_i' \geq 0$ for $y = y_j$, then $\eta_i \eta_i' > 0$ for $y > y_j$ [9]. Using diagonalization we find a solution of Eq. (6) in the form

$$b_1^\nu = \frac{P_1^\nu}{v_1}, \; a_i^\nu = \left(S_{i-1}^\nu v_i + P_i^\nu K_{ii-1} G_{i-1}\right) \frac{1}{Q_i},$$
$$b_i^\nu = \left(P_i^\nu g_i - S_{i-1}^\nu K_{i+1i} V_{i+1}\right) \frac{1}{Q_i}, \quad a_{n+1}^\nu = \frac{S_n^\nu}{g_{n+1}},$$

where $Q_i = g_i v_i + K_{ii-1} K_{i+1i} V_{i+1} G_{i-1} > 0$; v_i, V_i, P_i^ν, g_i, G_i, S_i^ν are expressed in terms of the recursive relationships

$$v_i = \xi_{ii} K_{i+1i} V_{i+1} + K_{ii} \xi_{ii}' v_{i+1} \eta_{i+1i} > 0,$$
$$V_i = \eta_{ii-1}' v_i + K_{i+1i} V_{i+1} > 0, \quad v_{n+1} = 1, \quad V_{n+1} = \eta_{n+1n}',$$
$$P_i^\nu = r_i^\nu K_{i+1i} V_{i+1} + \eta_{i+1i} \left(R_i^\nu v_{i+1} + K_{i+1i} P_{i+1}^\nu\right), \quad P_{n+1}^\nu = 0$$

for $i = n, \ldots, 1$;

$$g_{i+1} = K_{i+1i} \eta_{i+1i}' G_i + K_{ii} \xi_{ii}' \eta_{i+1i} g_i > 0,$$
$$G_i = \xi_{ii} g_i + K_{ii-1} G_{i-1} > 0, \quad g_1 = G_1 = 1, \quad S_i^\nu = R_i^\nu G_i -$$
$$- K_{ii} \xi_{ii}' \left(r_i^\nu g_i - S_{i-1}^\nu\right), \quad S_0^\nu = 0$$

for $i = 1(2), \ldots, n(n+1)$. Considering the sufficient smoothness of the functions F_i, it can be shown that the integrals of Eq. (3) converge and permit differentiation the needed number of times.

After k-fold integration of the functions of Eq. (3) over x we find the flow potentials ϕ_i in the form

$$\varphi_i = \underbrace{\int \ldots \int}_{k} u_i \, dx \ldots dx + \operatorname{Re} \sum_{p=0}^{k} C_{ip} Z_i^p (0, z). \tag{8}$$

Here $Z_i^p(0, z)$ are formal Bers powers [8], with characteristic $K_i(y)$, for which the coefficients C_{ip} are defined by the character of the potential singular points at infinity. If we write the permeability function as $K_i(y) = (\beta_i y + \gamma_i)^2$, then f_i can be expressed in terms of the singular points of the harmonic functions $\Phi_i(x, y)$; $f_i = \Phi_i/\sqrt{K_i}$, while the solutions of the Cauchy problem of Eqs. (4), (5) take on the following form

$$\xi_1 = \sqrt{\frac{K_{11}}{K_1(y)}} e^{-\lambda(y - y_1)}, \quad \xi_i = -\sqrt{\frac{K_{ii-1}}{K_i(y)}} \operatorname{sh} \lambda (y - y_{i-1}), \tag{9}$$
$$\eta_i = \frac{\beta_i \operatorname{sh} \lambda (y - y_i) + \lambda \sqrt{K_{ii}} \operatorname{ch} \lambda (y - y_i)}{\lambda \sqrt{K_i(y)}}, \quad \eta_{n+1} = \sqrt{\frac{K_{n+1n}}{K_{n+1}(y)}} e^{\lambda(y - y_n)},$$

i.e., in the given case the potentials are expressible in quadratures of Eq. (8).

In the expressions obtained it is possible to transform to the limit as $\ell_\mu = y_{\mu-1} - y_\mu \to 0$, $K_\mu \to \infty$ ($K_\mu \to 0$) for arbitrary homogeneous zones D_μ. In this case the layers D_μ degenerate into crevices (screens) characterized respectively by the parameters $A_\mu = \lim \ell_\mu K_\mu$, $B_\mu = \lim(\ell_\mu / K_\mu)$.

We will illustrate the above with the example of flow beheath a point dike in layered-homogeneous soil consisting of three zones D_i, of which the middle zone D_2 degenerates into a crevice or screen. In the given case $f_1 = (H_1 - H_2)\pi^{-1} \tan^{-1} x/y$ (H_i are values of the potential ϕ_1 on the water line $y = 0$, $y_i < 0$), while for the function ξ_1 instead of condition (5) we have $\xi_1 = 0$ at $y = 0$. Since $f_1 \in L(\partial D_1)$, then after differentiation of f_1 with respect to x with consideration of Eq. (9), where $\beta_i = 0$, $\xi_1 = \sinh \lambda y/\sinh \lambda y_1$, we find the functions u_i of Eq. (3), whence the flow potentials finally take on the form of Eq. (8),

$$\varphi_1 = \frac{H_1 - H_2}{\pi}\left[\operatorname{arctg}\frac{x}{y} - \int_0^\infty \frac{a}{\lambda Q} e^{\lambda y_1} \operatorname{sh}\lambda y \sin \lambda x \, d\lambda\right] + \frac{H_1 + H_2}{2},$$

$$\varphi_3 = K_1 \frac{H_2 - H_1}{\pi} \int_0^\infty \frac{1}{\lambda Q} e^{\lambda(y - y_1)} \sin \lambda x \, d\lambda + \frac{H_1 + H_2}{2}.$$

Here $a = K_3 - K_1 + \lambda A$, $Q = (K_3 + \lambda A)s + K_1 c$ for a fissure with parameter A and $a = K_3 - K_1 - K_1 K_3 \lambda B$, $Q = K_3 s + K_1 c(K_3 \lambda B + 1)$ for a screen with parameter B; $s = \sinh \lambda |y_1|$; $c = \cosh \lambda y_1$. Hence, in particular, it follows that a horizontal fissure $y = y_1$ increases, while a screen $y = y_1$ decreases the filtration rate on the water line, with the potential at the fissure and the flow at the screen being continuous, which agrees with other fissure models [6].

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